

MAXIMAL ABELIAN SUBALGEBRAS OF \mathcal{O}_n

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Abstract

We consider maximal abelian subalgebras of \mathcal{O}_n which are invariant to the standard circle action. It turns out that these are all contained in the zero grade of \mathcal{O}_n . Then we consider shift invariant maximal abelian subalgebras of the zero grade, which are also invariant to a “second shift” map, and show that these are just infinite tensor products of diagonal matrices in the standard UHF picture of the zero grade.

1 Introduction

In this note, we are concerned with certain abelian subalgebras of Cuntz algebra \mathcal{O}_n . Let $\mathcal{O}_n = C^*(s_1, \dots, s_n)$. As usual, for $\mu = i_1 \dots i_k$, $i_j \in \{1, \dots, n\}$, we let $|\mu| = k$ be the length of μ and denote $s_{i_1} \dots s_{i_k}$ by s_μ . The set of all finite words in $\{1, \dots, n\}$ is denoted by $\mathcal{W}(n)$. Let

$$\sigma(x) = \sum_{i=1}^n s_i x s_i^* \quad (1)$$

be the canonical endomorphism on \mathcal{O}_n ,

$$\omega_t(s_i) = t s_i, \quad i = 1, \dots, n, \quad t \in \mathbb{T} \quad (2)$$

the standard circle action, and consider the C^* -subalgebra \mathcal{D} of \mathcal{O}_n defined as

$$\mathcal{D} = C^*\{s_\mu s_\mu^*, \mu \in \mathcal{W}(n)\} \quad (3)$$

Then \mathcal{D} is an abelian subalgebra (cf. [2]), and below we list some of its properties:

- $\sigma(\mathcal{D}) \subset \mathcal{D}$
- $\omega_t(\mathcal{D}) \subset \mathcal{D}$; in fact, even more is true: $\omega_t(d) = d$, for all $t \in \mathbb{T}$ and $d \in \mathcal{D}$; hence, $\mathcal{D} \subset \mathcal{O}_n^0$, the fixed-point algebra under the circle action
- \mathcal{D} is a maximal abelian subalgebra of \mathcal{O}_n^0 ; furthermore, \mathcal{D} is maximal abelian in \mathcal{O}_n (cf. [3])

The subject of the present work is to give a characterisation of \mathcal{D} in the above terms. More precisely, we prove (cf. 4.4):

Theorem 1.1 *Let A be a maximal abelian subalgebra of \mathcal{O}_n such that $\omega(A) \subset A$, $\sigma(A) \subset A$ and $\tilde{\sigma}(A) \subset A$. Then there is an automorphism α_U of \mathcal{O}_n , determined by a unitary $U \in M_n(\mathbb{C})$, such that $\alpha_U(A) = \mathcal{D}$. Furthermore, α_U commutes with ω , σ and $\tilde{\sigma}$.*

We now describe the above notation. Let $\psi : \mathcal{O}_n \rightarrow M_n \otimes \mathcal{O}_n$ be given by

$$x \xrightarrow{\psi} \begin{bmatrix} s_1^* x s_1 & \dots & s_1^* x s_n \\ \vdots & & \vdots \\ s_n^* x s_1 & \dots & s_n^* x s_n \end{bmatrix} \quad (4)$$

Then ψ is an isomorphism from \mathcal{O}_n onto $M_n \otimes \mathcal{O}_n$ (cf. [1]).

The endomorphism $\tilde{\sigma} : \mathcal{O}_n \rightarrow \mathcal{O}_n$ is the “second shift”, given by the formula

$$\tilde{\sigma} = \sum_{i,j,k} s_i s_k s_i^* x s_j s_k^* s_j^*, \quad (5)$$

and determined by the following diagram

$$\begin{array}{ccc} \mathcal{O}_n & \xrightarrow{\psi} & M_n \otimes \mathcal{O}_n \\ \tilde{\sigma} \downarrow & & id \downarrow \otimes \sigma \\ \mathcal{O}_n & \xleftarrow{\psi^{-1}} & M_n \otimes \mathcal{O}_n \end{array} \quad (6)$$

Note that it follows easily that

$$\tilde{\sigma}(1) = 1, \quad \tilde{\sigma}(s_i s_j^*) = s_i s_j^*,$$

and

$$\tilde{\sigma}(s_i s_\mu s_\nu^* s_j^*) = s_i \sigma(s_\mu s_\nu^*) s_j^*,$$

for all $\mu, \nu \in \mathcal{W}(n)$ such that $|\mu| = |\nu|$.

Finally, for $u = [u_{ij}]_{i,j=1}^n$ a unitary in $M_n(\mathbb{C})$, let $U = \sum_{i,j} u_{ij} s_i s_j^*$. Then U is a unitary in \mathcal{O}_n , and the map $s_i \mapsto U s_i$, $i = 1, \dots, n$ extends to an isomorphism of \mathcal{O}_n , denoted α_U .

We also prove the same result as 4.4, but starting from slightly different assumptions (cf. 4.5):

Theorem 1.2 *Let A be a maximal abelian algebra in \mathcal{O}_n such that $A \cap \mathcal{O}_n^0$ is maximal abelian in \mathcal{O}_n^0 , $\sigma(A) \subset A$ and $\tilde{\sigma}(A) \subset A$. Then there is an automorphism α_U of \mathcal{O}_n , determined by a unitary $U \in M_n(\mathbb{C})$, such that $\alpha_U(A) = \mathcal{D}$. Furthermore, α_U commutes with σ and $\tilde{\sigma}$.*

The paper is organised as follows. In Section 2, we show that an algebra which is maximal in the class of abelian algebras that are invariant under the action of the circle is indeed maximal

abelian. In Section 3, we describe maximal abelian subalgebras that are invariant under two shift maps. Both Section 2 and 3 are done in a slightly more general setting. Finally, in Section 4, we apply these results to the particular case of \mathcal{O}_n and easily obtain the stated characterisation of the abelian subalgebra A .

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2 Maximal (abelian \mathbb{T} -invariant $*$ -subalgebras)

Take a C^* -algebra B , and let $\omega : \mathbb{T} \longrightarrow \text{Aut}(B)$ be a homomorphism that is continuous in the topology of pointwise convergence. This means that for each $b \in B$ the map $t \mapsto \omega_t(b)$ is continuous, and the triple (B, \mathbb{T}, ω) is called a C^* -dynamical system (cf. [4, 7.4.1]).

Consider the class of $*$ -subalgebras A of B which are abelian and \mathbb{T} -invariant (i.e. $\omega_t(a) \in A$ for all $a \in A$ and all $t \in \mathbb{T}$). We will call a subalgebra which is maximal in this class a maximal (abelian \mathbb{T} -invariant $*$ -subalgebra), using brackets to avoid ambiguity. Our task is to show that we can remove the brackets, that is, we show that a maximal (abelian \mathbb{T} -invariant $*$ -subalgebra) is actually maximal abelian.

The main result of this section is 2.7. For convenience we assume that B is a subalgebra of $B(H)$ for some Hilbert space H .

Definition 2.1 Let (B, \mathbb{T}, ω) be a C^* -dynamical system, and define $B_n = \{b \in B : \omega_t(b) = t^n b, \text{ for all } t \in \mathbb{T}\}$. Let $\pi_n : B \rightarrow B$ be defined as

$$\pi_n(b) = \int_{\mathbb{T}} t^{-n} \omega_t(b) dt,$$

where dt is the Haar measure on \mathbb{T} (i.e. normalised Lebesgue measure). Then each B_n is a closed linear subspace in B , and each π_n is a linear contraction with image B_n . Also B_0 is a subalgebra and π_0 is a conditional expectation to B_0 . Furthermore, we have

$$\pi_m(b) = \delta_{n,m} b, \quad b \in B_n.$$

Proposition 2.2 *The commutant $A' \cap B$ of A in B is an \mathbb{T} -invariant $*$ -subalgebra of B , and $A \subset A'$. Further the image of $A' \cap B$ under π_n is contained in $A' \cap B$.*

Proof: Suppose that $b \in A' \cap B$. Since A is a $*$ -subalgebra, for all $a \in A$, $a^*b = ba^*$, so we see that

$b^*a = ab^*$. Likewise, for any $t \in \mathbb{T}$, $\omega_{t^{-1}}(a)b = b\omega_{t^{-1}}(a)$ so $a\omega_t(b) = \omega_t(b)a$. As A is abelian, $A \subset A'$. Finally we take the equation $a\omega_t(b) = \omega_t(b)a$, divide by t^n and integrate to see that $\pi_n(b) \in A'$. \square

Proposition 2.3 *The image of $A' \cap B$ under π_0 is contained in A .*

Proof: Suppose that $b \in A' \cap B$. By considering $b + b^*$ and $i(b - b^*)$ we may suppose that b is actually Hermitian. Then $\pi_0(b)$ is Hermitian and fixed by the circle action, so the algebra generated by A and $\pi_0(b)$ is an abelian circle invariant *-subalgebra of B , and so $\pi_0(b) \in A$ by maximality. \square

Proposition 2.4 *The image of $A' \cap B$ under π_n is contained in A for all $n \in \mathbb{Z}$.*

Proof: Suppose that $b \in A' \cap B$. Then $\pi_n(b) \in A'$, and $\pi_n(b)\pi_n(b)^*$ is an Hermitian circle invariant element of $A' \cap B$. By maximality we then have $\pi_n(b)\pi_n(b)^* \in A$, and similarly we have $\pi_n(b)^*\pi_n(b) \in A$. This means that $\pi_n(b)$ commutes with both $\pi_n(b)\pi_n(b)^*$ and $\pi_n(b)^*\pi_n(b)$, and so is normal by the next lemma. Now the algebra generated by A , $\pi_n(b)$ and $\pi_n(b)^*$ is an abelian circle invariant *-algebra, so $\pi_n(b) \in A$ by maximality. \square

We initially proved the next lemma using polar decomposition. The following much simpler proof is due independently to S. Wassermann and N-C. Wong:

Lemma 2.5 *Let $x \in B$ commute with both xx^* and x^*x . Then x is normal.*

Proof: We need to show that $xx^* - x^*x = 0$. Since $xx^* - x^*x$ is selfadjoint, that is equivalent to $(xx^* - x^*x)^2 = 0$, which follows immediately, since the assumption implies

$$x^*xx^* = x(x^*x)x^* \text{ and } xx^*x^*x = x^*(xx^*)x$$

\square

Proposition 2.6 *$A' \cap B \subset A''$.*

Proof: Take $b \in A' \cap B$. For any $\xi, \eta \in H$ we define a continuous function $f : \mathbb{T} \rightarrow \mathbb{C}$ by $f(t) = \langle \xi, \omega_t(b)(\eta) \rangle$. By Fourier analysis we get Fourier coefficients $f_n = \langle \xi, \pi_n(b)(\eta) \rangle$, where

$$\sum_{n=-m}^m t^n f_n \longrightarrow f(t), \quad t \in \mathbb{T} \quad (7)$$

in the $L^2(\mathbb{T})$ topology as $m \rightarrow \infty$. Since $B \subset B(H)$, we can put $\eta = c(\kappa)$ for some $c \in A'$ and $\kappa \in H$. Then as $\pi_n(b) \in A$ we see that $f_n = \langle \xi, c\pi_n(b)(\kappa) \rangle = \langle c^*\xi, \pi_n(b)(\kappa) \rangle$. Now we can write

$$\sum_{n=-m}^m t^n f_n \rightarrow \langle c^*\xi, \omega(b)(\kappa) \rangle = \langle \xi, c\omega(b)(\kappa) \rangle, \quad t \in \mathbb{T}$$

in the $L^2(\mathbb{T})$ topology as $m \rightarrow \infty$. The two limits are the same in $L^2(\mathbb{T})$, so $\langle \xi, c\omega_t(b)(\kappa) \rangle = \langle \xi, \omega_t(b)c(\kappa) \rangle$ almost everywhere in \mathbb{T} . By continuity they are the same at $t = 1$, so $cb = bc$. \square

Theorem 2.7 *Let (B, \mathbb{T}, ω) be a C^* -dynamical system, and suppose that A is a maximal (abelian \mathbb{T} -invariant $*$ -subalgebra) of B . Then A is a maximal abelian subalgebra of B .*

Proof: By the previous proposition we see that $A' \cap B$ is abelian. Now $A' \cap B$ is an abelian circle invariant $*$ -subalgebra of B which contains A , so $A = A' \cap B$ by the maximality condition on A . But the equation $A = A' \cap B$ means that A is maximal abelian in B . \square

The following example – due to R. Exel – shows that the previous theorem does not hold for an arbitrary dynamical system (B, G, ω) , even with G compact:

Example 2.8 Consider the adjoint action of SU_2 on $M_2(\mathbb{C})$. The subalgebra consisting of the complex multiples of the identity is maximal among the class of abelian SU_2 -invariant $*$ -subalgebras. However, it is not maximal abelian, as it is properly contained in the diagonal matrices.

3 Maximal abelian $*$ -subalgebras of \mathcal{O}_n contained in the zero grade

Let B be a unital C^* -algebra with a given isomorphism $\psi : B \rightarrow M_n \otimes B$ with $\psi(1) = I_n \otimes 1$, I_n being the identity matrix in M_n . We define isomorphisms $\psi_m : B \rightarrow (M_n)^{\otimes m} \otimes B$ ($m \geq 0$) recursively, beginning with $\psi_0 : B \rightarrow B$ the identity, $\psi_1 = \psi$, and continuing by defining ψ_{m+1} to be the composition

$$B \xrightarrow{\psi_m} (M_n)^{\otimes m} \otimes B \xrightarrow{\text{id}^{\otimes m} \otimes \psi} (M_n)^{\otimes m+1} \otimes B ,$$

where $\text{id} : M_n \rightarrow M_n$ is the identity map. Now we define an algebra map $\kappa_m : M_n^{\otimes m} \rightarrow B$ by $\kappa_m(x) = \psi_m^{-1}(x \otimes 1)$. Since $\psi(1) = I_n \otimes 1$ we get the commutative diagram

$$\begin{array}{ccc} M_n^{\otimes m} & \xrightarrow{\kappa_m} & B \\ \downarrow \text{id} \otimes I_n & & \downarrow \text{id}_B \\ M_n^{\otimes m+1} & \xrightarrow{\kappa_{m+1}} & B \end{array}$$

Define $C \subset B$ to be the closure of the union of the subalgebras $\kappa_m(M_n^{\otimes m})$.

We can define shift maps $\sigma_m : B \rightarrow B$ ($m \geq 1$) by the composition

$$B \xrightarrow{\psi_{m-1}^{-1}} (M_n)^{\otimes m-1} \otimes B \xrightarrow{\text{id}^{\otimes m-1} \otimes f} (M_n)^{\otimes m} \otimes B \xrightarrow{\psi_m^{-1}} B ,$$

where $f : B \rightarrow M_n \otimes B$ is the algebra map $f(b) = I_n \otimes b$.

Let $E_{ij} \in M_n$ be the matrix with entry 1 in row i column j , and zeros elsewhere. Define a linear map $e_{ij} : M_n \rightarrow \mathbb{C}$ by $e_{ij}(E_{kl}) = \delta_{ik}\delta_{jl}$. Now we can define a map $\chi_{mij} : B \rightarrow B$ ($m \geq 1$) by the composition

$$B \xrightarrow{\psi_m} (M_n)^{\otimes m} \otimes B \xrightarrow{\text{id}^{\otimes m-1} \otimes e_{ij} \otimes \text{id}_B} (M_n)^{\otimes m-1} \otimes B \xrightarrow{\psi_{m-1}^{-1}} B .$$

Proposition 3.1 For all $b \in B$ and $y \in M_n \otimes B$, $(e_{ij} \otimes \text{id}_B)(f(b).y) = b.((e_{ij} \otimes \text{id}_B)(y))$ and $(e_{ij} \otimes \text{id}_B)(y.f(b)) = ((e_{ij} \otimes \text{id}_B)(y)).b$.

Proof: Take $y = y_1 \otimes y_2 \in M_n \otimes B$ (linear combinations of terms of this form are dense in $M_n \otimes B$). Then

$$\begin{aligned} (e_{ij} \otimes \text{id}_B)(f(b).y) &= (e_{ij} \otimes \text{id}_B)((I \otimes b)(y_1 \otimes y_2)) = \\ (e_{ij} \otimes \text{id}_B)(y_1 \otimes by_2) &= e_{ij}(y_1)by_2 = b.((e_{ij} \otimes \text{id}_B)(y)) \end{aligned}$$

The other way round is the same. \square

Corollary 3.2 For all $b, c \in B$, $\chi_{mij}(\sigma_m(b).c) = b.\chi_{mij}(c)$ and $\chi_{mij}(c.\sigma_m(b)) = \chi_{mij}(c).b$.

Proof: This is essentially the same as the previous proposition. \square

Corollary 3.3 Suppose that A is a maximal abelian $*$ -subalgebra of B , obeying the condition $\sigma_m(A) \subset A$. Then for all $1 \leq i, j \leq n$, $\chi_{mij}(A) \subset A$.

Proof: Take $a \in A$. Then for all $a' \in A$ we have $\sigma_m(a').a = a.\sigma_m(a')$. Applying χ_{mij} to this we get $a'.\chi_{mij}(a) = \chi_{mij}(a).a'$, so $\chi_{mij}(a) \in A$ by maximality. \square

Proposition 3.4 Suppose that A is a maximal abelian $*$ -subalgebra of B , obeying the condition $\sigma_1(A) \subset A$. Then $\psi_m(A) \subset M_n^{\otimes m} \otimes A$.

Proof: First note that $\psi(a) = \sum_{ij} E_{ij} \otimes \chi_{1ij}(a)$, so $\psi(A) \subset M_n \otimes A$ by the last proposition. The rest follows by induction. \square

Definition 3.5 Take a unital algebra map $\phi : A \rightarrow \mathbb{C}$, and extend it to a positive contraction $\phi : B \rightarrow \mathbb{C}$. Then we define a map $\phi_m : B \rightarrow M_n^{\otimes m}$ by

$$B \xrightarrow{\psi_m} M_n^{\otimes m} \otimes B \xrightarrow{\text{id}^{\otimes m} \otimes \phi} M_n^{\otimes m}.$$

Since $\phi_m(1) = 1$, it follows from [4, 3.1.6] that ϕ_m is a contraction. On the other hand, this is clearly a unital homomorphism when restricted to A . We denote by D the image of $\phi_1 : A \rightarrow M_n$.

Proposition 3.6 If $\sigma_1(A) \subset A$, then $\phi_{m+1}(A) \subset M_n \otimes \phi_m(A)$.

Proof: We can write ϕ_{m+1} as

$$A \xrightarrow{\psi_1} M_n \otimes A \xrightarrow{\text{id} \otimes \psi_m} (M_n)^{\otimes m+1} \otimes A \xrightarrow{\text{id}^{\otimes m+1} \otimes \phi} M_n^{\otimes m+1},$$

which can be rewritten as

$$A \xrightarrow{\psi_1} M_n \otimes A \xrightarrow{\text{id} \otimes \phi_m} M_n^{\otimes m+1},$$

so we see that $\phi_{m+1}(A) \subset M_n \otimes \phi_m(A)$. \square

Proposition 3.7 *If $\sigma_2(A) \subset A$, then $(\text{id} \otimes e_{ij} \otimes \text{id}^{\otimes m-1})\phi_{m+1}(A) \subset \phi_m(A)$ for $m \geq 1$.*

Proof: The map $(\text{id} \otimes e_{ij} \otimes \text{id}^{\otimes m-1}) \circ \phi_{m+1}$ is

$$B \xrightarrow{\psi_{m+1}} M_n^{\otimes m+1} \otimes B \xrightarrow{\text{id}^{\otimes m+1} \otimes \phi} M_n^{\otimes m+1} \xrightarrow{\text{id} \otimes e_{ij} \otimes \text{id}^{\otimes m-1}} M_n^{\otimes m},$$

which can be rewritten as

$$B \xrightarrow{\psi_{m+1}} M_n^{\otimes m+1} \otimes B \xrightarrow{\text{id} \otimes e_{ij} \otimes \text{id}^{\otimes m-1} \otimes \text{id}_B} M_n^{\otimes m} \otimes B \xrightarrow{\psi_m^{-1}} B \xrightarrow{\psi_m} M_n^{\otimes m} \otimes B \xrightarrow{\text{id}^{\otimes m} \otimes \phi} M_n^{\otimes m}.$$

This can be shown to be

$$B \xrightarrow{\psi_2} M_n^{\otimes 2} \otimes B \xrightarrow{\text{id} \otimes e_{ij} \otimes \text{id}_B} M_n \otimes B \xrightarrow{\psi_1^{-1}} B \xrightarrow{\psi_m} M_n^{\otimes m} \otimes B \xrightarrow{\text{id}^{\otimes m} \otimes \phi} M_n^{\otimes m},$$

so we see that $(\text{id} \otimes e_{ij} \otimes \text{id}^{\otimes m-1}) \circ \phi_{m+1} = \phi_m \circ \chi_{2ij} : B \rightarrow M_n^{\otimes m}$. Now use $\chi_{2ij}(A) \subset A$. \square

Corollary 3.8 *If $\sigma_1(A) \subset A$ and $\sigma_2(A) \subset A$, then $\phi_m(A) \subset D^{\otimes m}$.*

Proof: This is proved by induction. First note that $\phi_1(A) \subset D^{\otimes 1}$ by definition of D . Now assume that $\phi_m(A) \subset D^{\otimes m}$, and consider $m+1$. By the previous proposition we see that $\phi_{m+1}(A) \subset D \otimes M_n \otimes D^{\otimes m-1}$, whereas the proposition before that says that $\phi_{m+1}(A) \subset M_n \otimes D^{\otimes m}$. The result follows by standard linear algebra. \square

Proposition 3.9 *Given $c \in C$ and $\epsilon > 0$, there is an $m \geq 1$ so that $|\kappa_m(\phi_m(c)) - c| < \epsilon$.*

Proof: There is an $m \geq 1$ and an $x \in M_n^{\otimes m}$ so that $|c - \kappa_m(x)| < \epsilon/2$. Since $\phi(1) = 1$ we get $\phi_m(\kappa_m(x)) = x$, and since ϕ_m is a contraction, $|\phi_m(c) - x| < \epsilon/2$. Finally as κ_m is a contraction, $|\kappa_m(\phi_m(c)) - \kappa_m(x)| < \epsilon/2$. \square

Now, let D^∞ stand for the closure of the union of $\kappa_m(D^{\otimes m})$ for $m \geq 1$. Then we have:

Theorem 3.10 *Suppose that $A \cap C$ is maximal abelian in C . Then $A \cap C = D^\infty$ and D is maximal abelian in $M_n(\mathbb{C})$.*

Proof: 3.8 and 3.9 show that $A \cap C \subset D^\infty$. Since D^∞ is abelian and $A \cap C$ is maximal, it follows that $A \cap C = D^\infty$. Hence, D is maximal abelian in $M_n(\mathbb{C})$. \square

4 Maximal (abelian \mathbb{T} -invariant $*$ -subalgebras) of \mathcal{O}_n

In this section, we apply the results from Section 2 and 3 to maximal abelian subalgebras of \mathcal{O}_n that are invariant under the standard circle action. The notation is as in the Introduction.

The next lemma is probably well-known, but we couldn't find a reference:

Lemma 4.1 *Let x be in \mathcal{O}_n^k (i.e. $\omega_t(x) = t^k x$), for $k \neq 0$. If x is normal, then $x = 0$.*

Proof: Suppose $k > 0$. Let $y = x(s_1^*)^k \in \mathcal{O}_n^0$, and let τ be the faithful normalised trace on $\mathcal{O}_n^0 \cong M_{n^\infty}(\mathbb{C})$. Then $yy^* = xx^*$, $y^*y = s_1^k x^* x (s_1^*)^k$, and $\tau(yy^*) = \tau(y^*y)$ imply

$$\tau(xx^*) = n^{-k} \tau(x^*x)$$

If $xx^* = x^*x$, then

$$\tau(x^*x) = n^{-k} \tau(x^*x),$$

hence $\tau(x^*x) = 0$. □

Theorem 4.2 *Suppose that A is a maximal (abelian \mathbb{T} -invariant $*$ -subalgebra) of \mathcal{O}_n . Then $A \subset \mathcal{O}_n^0$.*

Proof: By the previous lemma, $\pi_k(a) = 0$ for all $a \in A$ and $k \neq 0$. By Fourier analysis we get

$$\sum_{k=-m}^m t^k \langle \xi, \pi_k(a)(\eta) \rangle \rightarrow \langle \xi, \omega_t(a)(\eta) \rangle, \quad t \in \mathbb{T}$$

in the $L^2(\mathbb{T})$ topology as $m \rightarrow \infty$. But then $\langle \xi, \omega_t(a)(\eta) \rangle$ is constant on \mathbb{T} , so $\omega_t(a) = a$. □

Remark 4.3 Note that, if B is a C^* -algebra of the form $B = A \rtimes_\alpha \mathbb{N}$, where A has a normalised faithful trace, α is a trace-scaling endomorphism, and the circle action is just the dual action with respect to this crossed-product representation, the above argument shows that any maximal (abelian \mathbb{T} -invariant $*$ -subalgebra) will be contained in the fixed-point algebra for the circle action. This will be the case if B is a simple Cuntz-Krieger algebra, or more generally, for certain Cuntz-Pimsner algebras (cf. [5]).

Theorem 4.4 *Let A be a maximal abelian subalgebra of \mathcal{O}_n such that $\omega(A) \subset A$, $\sigma(A) \subset A$ and $\tilde{\sigma}(A) \subset A$. Then there is an automorphism α_U of \mathcal{O}_n , determined by a unitary $U \in M_n(\mathbb{C})$, such that $\alpha_U(A) = \mathcal{D}$. Furthermore, α_U commutes with ω , σ and $\tilde{\sigma}$.*

Proof: By 4.2, $A \subset \mathcal{O}_n^0$. The result then follows from 3.10, with $B = \mathcal{O}_n^0$, $\sigma = \sigma_1$ and $\tilde{\sigma} = \sigma_2$, while u is any unitary in $M_n(\mathbb{C})$ that diagonalises the maximal abelian subalgebra D . □

Theorem 4.5 *Let A be a maximal abelian algebra in \mathcal{O}_n such that $A \cap \mathcal{O}_n^0$ is maximal abelian in \mathcal{O}_n^0 , $\sigma(A) \subset A$ and $\tilde{\sigma}(A) \subset A$. Then there is an automorphism α_U of \mathcal{O}_n , determined by a unitary $U \in M_n(\mathbb{C})$, such that $\alpha_U(A) = \mathcal{D}$. Furthermore, α_U commutes with σ and $\tilde{\sigma}$.*

Proof: We apply 3.10, with $B = \mathcal{O}_n$, and σ and $\tilde{\sigma}$ as in the previous theorem. That shows that $D^\infty \subset A$. Since D^∞ is maximal abelian in \mathcal{O}_n (cf. [3, 2.18]), $A = D^\infty$. Note that this means that all of A is contained in the zero grade, although no assumptions on the circle action were made. \square

References

- [1] M. D. Choi, *A simple C^* -algebra generated by two finite-order unitaries*, Canad. J. Math 31 (1979), 867–880
- [2] J. Cuntz, *Simple C^* -algebras generated by isometries*, Comm. Math. Phys. 57 (1977), 173–185
- [3] J. Cuntz and W. Krieger, *A class of C^* -algebras and topological Markov chains*, Invent. Math. 56 (1980), 251–268
- [4] G. K. Pedersen, *C^* -algebras and their automorphism groups*, London Mathematical Society Monographs, 14. Academic Press, 1979
- [5] M. Pimsner, *A class of C^* -algebras generalizing both Cuntz–Krieger algebras and crossed products by \mathbb{Z}* , Fields Inst. Commun., Amer. Math. Soc. (1997), no. 12, 189–212

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